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ON THE HOMOGENEOUS BIQUADRATIC DIOPHANTINE EQUATION WITH FIVE UNKNOWNS

$$x^4 - y^4 = 5(z^2 - w^2)T^2$$

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ABSTRACT

The biquadratic Diophantine equation with five unknowns represented by $x^4 - y^4 = 5(z^2 - w^2)T^2$ is analysed for finding its non-zero distinct integral solutions. Introducing the linear transformations $x = u + v, y = u - v, z = 2u + v, \mathrm{w} = 2u - v$ and employing the method of factorization different patterns of non zero distinct integer solutions of the equation under the above equation are obtained. A few interesting relations between the integral solutions and the special numbers namely Polygonal numbers, Pyramidal numbers, Centered Polygonal numbers, Centered Pyramidal numbers, Thabit-ibn-Kurrah number, Star number,Gnomonic number are exhibited.

Keywords- *Biquadratic with five unknowns, Integral solutions, Special numbers.*

2010 Mathematics Subject Classification:11D25

NOTATIONS:

Special Numbers	**Notations**	**Definitions**
Regular Polygonal number	$t_{m,n}$	$n(1 + \frac{(n-1)(m-2)}{2})$
Pronic Number	P_n	$n(n+1)$
Gnomonic Number	G_n	$2n-1$
Pentagonal Pyramidal Number	P_n^5	$\frac{n^2(n+1)}{2}$
Star Number	S_n	$6n(n-1)+1$
Woodall Number	W_n	$n2^n - 1$
Thabit-ibn-Kurrah Number	TK_n	$3(2^n)-1$
Centered Hexadecagonal Number	$Ct_{16,n}$	$8n(n+1)+1$
Stella Octangula number `	SO_n	$n(2n^2-1)$
Kynea number	Ky_n	$(2n+1)^2 - 2$
Centered hexagonal Pyramidal number	CP_n^6	$-n^3$

I. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular biquadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-2]. In this context one may refer [3-10] for various problems on the biquadratic Diophantine equations. However, often we come across homogeneous biquadratic equations and as such one may require its

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integral solution in its most general form. This paper concern with the homogeneous biquadratic equation with five unknowns $x^4 - y^4 = 5(z^2 - w^2)T^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The biquadratic diophantine equation with five unknowns to be solved for getting non-zero integral solution is

$$x^4 - y^4 = 5(z^2 - w^2)T^2 \qquad (1)$$

On substituting the linear transformations

$$x = u + v, y = u - v, z = 2u + v, \mathrm{w} = 2u - v \qquad (2)$$

in (1), it leads to

$$u^2 + v^2 = 5T^2 \qquad (3)$$

we present below different methods of solving (3) and thus obtain different pattern of integral solutions to (1).

2.1.Pattern I

Write $u^2 + v^2 = (u + iv)(u - iv)$ (4)

Assume $T(a,b) = a^2 + b^2 = (a + ib)(a - ib)$ where $a, b \neq 0$ (5)

Write 5 as $5 = (2 + i)(2 - i)$ (6)

Substituting (4),(5) and (6) in (3), we get

$$(u + iv)(u - iv) = \left[(2 + i)(2 - i)\right](a + ib)^2 (a - ib)^2$$

Equating the real and imaginary parts, the values of v and u are given by,

$$\left.\begin{aligned} u = u(a,b) = 2a^2 - 2b^2 - 2ab \\ v = v(a,b) = a^2 - b^2 + 4ab \end{aligned}\right\} \qquad (7)$$

Substituting these values in (2), we get

$$x(a,b) = 3a^2 - 3b^2 + 2ab$$

$$y(a,b) = a^2 - b^2 - 6ab$$

$$z(a,b) = 5a^2 - 5b^2$$

$$w(a,b) = 3a^2 - 3b^2 - 8ab$$

Thus, these values of x, y, z, w, T represent non-zero distinct integer solutions of (1).

Properties

1. $x(a,t_{3,a}) + 3y(a,t_{3,a}) - 2w(a,t_{3,a}) = 0$
2. $x^3(a,t_{3,a}) + 27y^3(a,t_{3,a}) - 8w^3(a,t_{3,a}) = -18x(a,t_{3,a})y(a,t_{3,a})w(a,t_{3,a})$
3. $3x(a,t_{3,a}) + 5y(a,t_{3,a}) = z(a,t_{3,a}) + 3w(a,t_{3,a})$
4. $3x(a,t_{3,a}) + y(a,t_{3,a}) - z(a,t_{3,a}) = 0$
5. $3w(a,t_{3,a}) - 4y(a,t_{3,a}) - z(a,t_{3,a}) = 0$
6. $3\left[z(a,t_{3,a}) - 2y(a,t_{3,a})\right] = 5w(a,t_{3,a}) - x(a,t_{3,a})$
7. $27z^3(a,t_{3,a}) - 64x^3(a,t_{3,a}) - w^3(a,t_{3,a}) = 36x(a,t_{3,a})z(a,t_{3,a})w(a,t_{3,a})$

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8. $y(a,t_{3,a})\left[27z^3(a,t_{3,a})-64x^3(a,t_{3,a})-w^3(a,t_{3,a})\right]=x(a,t_{3,a})\left[27w^3(a,t_{3,a})-64y^3(a,t_{3,a})-z^3(a,t_{3,a})\right]$

9. $w(a,t_{3,a})\left[27x^3(a,t_{3,a})+y^3(a,t_{3,a})-8z^3(a,t_{3,a})\right]=z(a,t_{3,a})\left[x^3(a,t_{3,a})+27y^3(a,t_{3,a})-8w^3(a,t_{3,a})\right]$

2.1.1. REMARK

Write (6) as

$$5=\begin{cases}(1+2i)(1-2i)\\ \dfrac{(2+11i)(2-11i)}{25}\\ \dfrac{(2+29i)(2-29i)}{169}\end{cases}$$

Following the above procedure, the other choices of solutions to (1) are obtained.

2.2. Pattern II

Equation (3) can be written as

$$u^2+v^2=5T^2\times 1 \tag{8}$$

Write 1 as

$$1=\frac{(m^2-n^2+2imn)(m^2-n^2-2imn)}{(m^2+n^2)^2} \tag{9}$$

Substituting (5), (6)and (9) in (8) and using the method of factorization , define

$$u+iv=(2+i)\frac{(m^2-n^2+i2mn)}{m^2+n^2}(a+ib)^2 \tag{10}$$

Equating real and imaginary parts of (10), we have

$$\left.\begin{aligned}u=u(a,b)&=\frac{1}{p^2+q^2}\left[2(a^2-b^2)(p^2-q^2)-8abpq-2ab(p^2-q^2)-2pq(a^2-b^2)\right]\\ v=v(a,b)&=\frac{1}{p^2+q^2}\left[(a^2-b^2)(p^2-q^2)-4abpq+4ab(p^2-q^2)+4pq(a^2-b^2)\right]\end{aligned}\right\} \tag{11}$$

To find the integral solutions of (1), substitute $a=(p^2+q^2)\mathrm{A}, b=(p^2+q^2)B$ in (11)and (5)

$$u(A,B)=2(p^2-q^2)(A^2-B^2)-8ABpq(p^2+q^2)-2(p^4-q^4)AB-2pq(A^2-B^2)$$
$$\therefore\ v(A,B)=(p^2-q^2)(A^2-B^2)-4ABpq(p^2+q^2)+4(p^4-q^4)AB+4pq(A^2-B^2)$$
$$T(A,B)=(p^2+q^2)^2A^2+(p^2+q^2)^2B^2$$

Then the corresponding integral values of x,y,z,w,T satisfying (1) are obtained as

$$x(p,q,A,B)=(p^4-q^4)\left[3(A^2-B^2)+2AB\right]+2pq(p^2+q^2)\left[A^2-B^2-6AB\right]$$
$$y(p,q,A,B)=(p^4-q^4)\left[A^2-B^2-6AB\right]+2pq(p^2+q^2)\left[A^2-B^2-2AB\right]$$
$$z(p,q,A,B)=5(p^4-q^4)\left[A^2-B^2\right]-20ABpq(p^2+q^2)$$
$$w(p,q,A,B)=(p^4-q^4)\left[3(A^2-B^2)-8AB\right]+2pq(p^2+q^2)\left[A^2-B^2-2AB\right]$$
$$T(p,q,A,B)=(p^2+q^2)^2\left[A^2+B^2\right]$$

Properties:

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1. $x(p,q,A,A)-3y(p,q,A,A)=20(p^4-q^4)t_{4,A}$
2. $T(p,q,A,A-1)=(p^2+q^2)^2\left[2P_{A-1}+1\right]$
3. $z(p,q,A,A)+w(p,q,A,A)-40(q^4-p^4)t_{4,A}\equiv 0(\bmod 20)$

2.2.1 REMARK

Also, 1 can be written as either

$$1=\frac{(1+i)^{2n}(1-i)^{2n}}{2^{2n}}$$

or $1=i^n(-i)^n$

Following the procedure as presented in Pattern II then the corresponding non-zero integral solutions satisfying (1) are obtained as

$$x(n,a,b)=(3a^2-3b^2+2ab)\cos\frac{n\pi}{2}+(a^2-b^2-6ab)\sin\frac{n\pi}{2}$$

$$y(n,a,b)=(a^2-b^2-6ab)\cos\frac{n\pi}{2}+(-3a^2+3b^2-2ab)\sin\frac{n\pi}{2}$$

$$z(n,a,b)=(5a^2-5b^2)\cos\frac{n\pi}{2}+(-10ab)\sin\frac{n\pi}{2}$$

$$w(n,a,b)=(3a^2-3b^2-8ab)\cos\frac{n\pi}{2}+(-4a^2+4b^2-6ab)\sin\frac{n\pi}{2}$$

$$T(a,b)=a^2+b^2$$

2.3.Pattern III

Equation (3) can be written as

$$1\times u^2=5T^2-v^2 \qquad (12)$$

Assume $u(a,b)=5a^2-b^2=(\sqrt{5}a+b)(\sqrt{5}a-b)$ (13)

$$1=(\sqrt{5}+2)(\sqrt{5}-2) \qquad (14)$$

Substituting (13) and (14) in (12) and using the method of factorization, define

$$(\sqrt{5}+2)(\sqrt{5}a+b)^2=\sqrt{5}T+v$$

Equating rational and irrational parts, we get

$$T(a,b)=5a^2+b^2+4ab$$

$$v(a,b)=10a^2+2b^2+10ab$$

Substituting the values of u,v in (2) the non-zero integral distinct points satisfying (1) are given by

$$x(a,b)=15a^2+b^2+10ab$$

$$y(a,b)=-5a^2-3b^2-10ab$$

$$z(a,b)=20a^2+10ab$$

$$w(a,b)=-4b^2-10ab$$

$$T(a,b)=5a^2+b^2+4ab$$

Properties

1. $2x(a,b)+2y(a,b)-z(a,b)-w(a,b)=0$

2. $z(a,a+1)-T(a,a+1)-w(a,a+1)=Ky_a+1$

3. $z(a,\frac{a^2+1}{2})+5w(a,\frac{a^2+1}{2})+80CP_a^3=0$

4. $x(b(b+1))-3T(b(b+1))+2CP_b^6=0$

5. $z(a,1)+5w(a,1)-10(2P_{a-1}-G_a+1)=0$

6. $6\left[y(b,b)-T(b,b)\right]$ is a nasty number.

2.3.1.REMARK

Note that, in addition to (14), one may represent 1 in the following ways:

$$1=\begin{cases}\dfrac{(\sqrt{5}+1)(\sqrt{5}-1)}{4}\\ \dfrac{(5\sqrt{5}+2)(5\sqrt{5}-2)}{121}\\ \dfrac{(5\sqrt{5}+11)(5\sqrt{5}-11)}{4}\\ \dfrac{(5\sqrt{5}+11)(5\sqrt{5}-11)}{4}\\ \dfrac{(13\sqrt{5}+2)(13\sqrt{5}-2)}{29^2},\dfrac{(13\sqrt{5}+19)(13\sqrt{5}-19)}{22^2},\dfrac{(13\sqrt{5}+29)(13\sqrt{5}-29)}{2^2}\\ \dfrac{(17\sqrt{5}+1)(17\sqrt{5}-1)}{38^2},\dfrac{(17\sqrt{5}+22)(17\sqrt{5}-22)}{31^2},\\ \dfrac{(17\sqrt{5}+31)(17\sqrt{5}-31)}{22^2},(17\sqrt{5}+38)(17\sqrt{5}-38)\end{cases}$$

Then, proceeding as in the above procedure , other choices of non-zero distinct integer solutions to (1) are obtained.

2.4.Pattern IV

Equation (3) can be written as

$$v^2=5T^2-u^2 \qquad (15)$$

Introducing the linear transformations

$$T=X+S;u=X+5S \qquad (16)$$

in (15), it is written as

$$v^2+20S^2=(2X)^2 \qquad (17)$$

which is satisfied by

$$S=2\alpha\beta,v=20\alpha^2-\beta^2,X=\frac{1}{2}\left[20\alpha^2+\beta^2\right]$$

To find integer solution replacing α by 2α and β by 2β, we get

$$X=40\alpha^2+2\beta^2,S=8\alpha\beta \qquad (18)$$

$$v=80\alpha^2-4\beta^2 \qquad (19)$$

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From (18) and (16) the values of T and u are

$$\left.\begin{aligned} T(\alpha,\beta) &= 40\alpha^2 + 2\beta^2 + 8\alpha\beta \\ u(\alpha,\beta) &= 40\alpha^2 + 2\beta^2 + 40\alpha\beta \end{aligned}\right\} \quad (20)$$

Substituting the values of u, v from (19)and (20) in (2) the corresponding non-zero integral solutions satisfying (1) are obtained as

$$x(\alpha,\beta) = 120\alpha^2 - 2\beta^2 + 40\alpha\beta$$
$$y(\alpha,\beta) = -40\alpha^2 + 6\beta^2 + 40\alpha\beta$$
$$z(\alpha,\beta) = 160\alpha^2 + 80\alpha\beta$$
$$w(\alpha,\beta) = 8\beta^2 + 80\alpha\beta$$
$$T(\alpha,\beta) = 40\alpha^2 + 2\beta^2 + 8\alpha\beta$$

Properties

1. $x(\alpha,\beta) - y(\alpha,\beta) - z(\alpha,\beta) + w(\alpha,\beta) \equiv 0 \pmod{96}$
2. $y(\alpha 2^\alpha, 1) + T(\alpha 2^\alpha, 1) - w(\alpha 2^\alpha, 1) + 32(W_\alpha + 1) = 0$
3. $z(\alpha,\beta) - x(\alpha,\beta) + y(\alpha,\beta) - w(\alpha,\beta) = 0$
4. $z(\alpha^2, \alpha+1) - 4T(\alpha^2, \alpha+1) + w(\alpha^2, \alpha+1) = 256P_\alpha^5$
5. $3x(\alpha,\alpha) + y(\alpha,\alpha) - 2z(\alpha,\alpha) = 0$

2.4.1. REMARK

It is to be noted that instead of (16) one may consider the linear transformation

$$T = X - S; u = X - 5S$$

For this choice, the corresponding non-zero distinct integral solutions to (1) are represented below,

$$x(\alpha,\beta) = 120\alpha^2 - 2\beta^2 - 40\alpha\beta$$
$$y(\alpha,\beta) = -40\alpha^2 + 6\beta^2 - 40\alpha\beta$$
$$z(\alpha,\beta) = 160\alpha^2 - 80\alpha\beta$$
$$w(\alpha,\beta) = 8\beta^2 - 80\alpha\beta$$
$$T(\alpha,\beta) = 40\alpha^2 + 2\beta^2 - 8\alpha\beta$$

2.5.Pattern V

Equation (3) is written as

$$(2T + v)(2T - v) = (u + T)(u - T) \quad (21)$$

which is expressed in the form of ratio as,

$$\frac{u + T}{2T + v} = \frac{2T - v}{u - T} = \frac{\alpha}{\beta}, \beta \neq 0 \quad (22)$$

This is equivalent to the following two equations

$$\beta u - \alpha v + (\beta - 2\alpha)T = 0$$
$$\alpha u + \beta v - (\alpha + 2\beta)T = 0$$

Applying the method of cross multiplication, we get

$$u(\alpha,\beta) = \alpha^2 - \beta^2 + 4\alpha\beta$$

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$v(\alpha,\beta) = 2\alpha\beta - 2\alpha^2 + 2\beta^2$

$T(\alpha,\beta) = \alpha^2 + \beta^2$

Substituting the values of u, v in (2) the non-zero distinct integral points satisfying (1) are given by

$x(\alpha,\beta) = -\alpha^2 + \beta^2 + 6\alpha\beta$

$y(\alpha,\beta) = 3\alpha^2 - 3\beta^2 + 2\alpha\beta$

$z(\alpha,\beta) = 10\alpha\beta$

$w(\alpha,\beta) = 4\alpha^2 - 4\beta^2 + 6\alpha\beta$

$T(\alpha,\beta) = \alpha^2 + \beta^2$

Properties

1. $6\left[y(\alpha,\beta) + 2T(\alpha,\beta) + x(\alpha,\beta) - w(\alpha,\beta)\right]$ is a nasty number.
2. $z(\alpha^2,\alpha(\alpha+1)) - x(\alpha^2,\alpha(\alpha+1)) - 2y(\alpha^2,\alpha(\alpha+1)) = 5\left[SO_\alpha + P_\alpha\right]$
3. $3x(2^\alpha,1) + y(2^\alpha,1) - TK_\alpha - 1 \equiv 0 \pmod{17}$
4. $w(\alpha(\alpha-1),\beta) + 3T(\alpha(\alpha-1),\beta) - x(\alpha(\alpha-1),\beta) - S_\alpha + 3 \equiv 0 \pmod 8$
5. $w(\alpha,(\alpha-1)) + 4T(\alpha,(\alpha-1)) - z(\alpha,(\alpha-1)) = ct_{16,\alpha} + 4P_\alpha - 1$

III. CONCLUSION

It is worth to mention to note that in (2) the transformations for z and w may be considered as $z = 2uv+1, w = 2uv-1$ and $z = uv+2, w = uv-2$. For these cases, the values of x, y, T are the same as above where as the values of z and w changes for every pattern. To conclude one may consider biquadratic equation with multivariables (≥5) and search for the their non-zero distinct integer solutions along with their corresponding properties.

REFERENCES

1. *Carmichael R.D., The Theory of numbers and Diophantine analysis, Dover publications, new York, 1959.*
2. *Dickson L.E., History of theory of Numbers, Chelsa Publishing company, New Yors, 1952.*
3. *Gopalan.M.A., and Pandichelvi.V., On the solutions of the Biquadratic equation* $\left(x^2-y^2\right)^2 = \left(z^2-1\right)^2 + w^4$ *, International conference on Mathematical methods and Computation,Pg:24-25, july 2009.*
4. *Gopalan.M.A., and Sangeetha. G., Integral solutions of Ternary Quartic equation* $x^2 - y^2 + 2xy = z^4$ *, Antartica J.Math., 7(1), 95-101, 2010.*
5. *Gopalan.M.A., and Sangeetha. G., Integral solutions of non-homogeneous Quartic equation* $x^4 - y^4 = \left(2\alpha^2 + 2\alpha + 1\right)\left(z^2 - w^2\right)$ *, impact J.Sci.Tech., 4(3), (July-Sep), 15-21, 2010.*
6. *Gopalan.M.A., and Sangeetha. G., Integral solutions of Non-homogeneous biquadratic* $x^4 + x^2 + y^2 - y = z^2 + z$ *, Acta Ciencia Indica, Vol.XXXVII M.No.4, 799-803, 2011.*
7. *Gopalan .M.A., and Sivkami.B., Integral solutions of quartic equation with four unknowns* $x^3 + y^3 + z^3 = 3xyz + 2(x+y)w^3$ *, Antartica J.Math, 10(2), 151-159, 2013.*

8. *Manju Somanath, Sangeetha. G., and Gopalan.M.A., Integral solutions of biquadratic equation with four unknowns given by* $xy+\left(k^2+1\right)z^2=5w^2$ *, Pacific-Asian Journal of Mathematics, 6(2),185-190, July- Dec 2012.*
9. *Manju Somanath, Sangeetha. G., and Gopalan.M.A., Integral solutions of non-homogeneous Quartic equation* $x^4-y^4=\left(k^2+1\right)\left(z^2-w^2\right)$*, Archimedes J.Math., 1(1), 51-57, 2011.*
10. *Sangeetha. G., Manju Somanath., Gopalan.M.A., and Pushparani., Integral solutions of the homogeneous biquadratic equation with five unknowns* $\left(x^3+y^3\right)z=\left(w^2-p^2\right)R^2$ *, International conference on Mathematical methods and Computation,Pg:221-226, Feb 2014.*